Derivation of Sum of Squares Formula

Let's say you've forgotten the formula for the sum of squares:

$$\sum_{i=1}^{n} i^2$$

But you need it desperately! How can you derive it for yourself?

First, rewrite the meaning of the formula in long-hand:

$$\begin{array}{rcccccccccc}
\sum_{i=1}^{n} i^2 & = & 1\cdot 1 & + & 2\cdot 2 & + & 3\cdot 3 & \ldots & n\cdot n \\
 & = & 1 & + & 2 & + & 3 & \ldots & n \\
 & & & + & 2 & + & 3 & \ldots & n \\
 & & & & & + & 3 & \ldots & n \\
 & & & & & & \ddots & & \\
 & & & & & & & + & n \\
 & = & \multicolumn{7}{l}{\sum_{i=1}^{n} \sum_{j=i}^{n} j}
\end{array}$$

So we just change our thinking from multiplication to summation! Now we have this nested summation, but that shouldn't be that horrid as it is linear inside. . .

Let's see. Let's focus on the inner sum first:

$$\begin{aligned}
\sum_{j=i}^{n} j &= \sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j && \text{forgot the short-hand formula for this, too} \\
&= \frac{n(n+1)}{2} - \frac{(i-1)(i-1+1)}{2} \\
&= \frac{n(n+1)}{2} - \frac{i(i-1)}{2}
\end{aligned}$$

Not bad. Now let's put it back into the outer sum and see what's up there:

$$\begin{aligned}
\sum_{i=1}^{n} \left(\frac{n(n+1)}{2} - \frac{i(i-1)}{2} \right) &= \frac{n^2(n+1)}{2} - \frac{1}{2} \sum_{i=1}^{n} i(i-1) && \text{factored out } n\text{'s from first sum since they were independent of the index of summation} \\
&= \frac{n^2(n+1)}{2} - \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i \\
\sum_{i=1}^{n} i^2 &= \frac{n^2(n+1)}{2} + \frac{n(n+1)}{4} - \frac{1}{2} \sum_{i=1}^{n} i^2
\end{aligned}$$

Ah crap! We're back to where we started!? How can we possibly go on?!

Let's see, we have the same thing on both sides of our equation, can't we add the left one to both sides to collect those terms together? That is, look at our last equation as:

$$A = \frac{n^2(n+1)}{2} + \frac{n(n+1)}{4} - \frac{1}{2} A$$

Now add $\frac{1}{2}A$ to both sides to get:

$$\frac{3}{2} A = \frac{n^2(n+1)}{2} + \frac{n(n+1)}{4}$$

And working from there we see that:

$$\begin{aligned}
\frac{3}{2}A &= \frac{n^2(n+1)}{2} + \frac{n(n+1)}{4} \\
&= \frac{2n^2(n+1) + n(n+1)}{4} \\
&= \frac{(2n^2+n)(n+1)}{4} \\
&= \frac{n(2n+1)(n+1)}{4} \\
A &= \frac{n(2n+1)(n+1)}{6}
\end{aligned}$$

And substituting back for A we get:

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

Voilà!